

THE VORONOI CONJECTURE FOR PARALLELOHEDRA WITH SIMPLY CONNECTED δ -SURFACE

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ABSTRACT. We show that Voronoi conjecture on parallelohedra is true for a new class of polytopes. Namely we show that if the surface of a parallelohedron remains simply connected after removing closed non-primitive faces of codimension 2 then for this polytope exists an affine transformation into Dirichlet-Voronoi domain of some lattice. Also we give more powerful condition using certain cycles on the surface of a parallelohedron with deleted faces.

1. VORONOI CONJECTURE

A d -dimensional polytope P is called a *parallelohedron* if we can tile \mathbb{R}^d into non-overlapping parallel copies of P . A correspondent tiling is called *face-to-face* if the intersection of any two copies of P is a face of both, otherwise it is called a *non face-to-face tiling*. McMullen [5] proved that if there is a non face-to-face tiling into copies of P then there is also a face-to-face tiling. It is clear that the face-to-face tiling for a given parallelohedron P is unique up to a translation. We will denote this tiling as $\mathcal{T}(P)$ or just \mathcal{T} when the generating polytope of the tiling is obvious.

In 1897 H. Minkowski [6] proved that every parallelohedron P is centrally symmetric, all facets of P are centrally symmetric, and projection of P along any its faces of codimension 2 is two-dimensional parallelohedron, i.e. parallelogram or centrally symmetric hexagon. Later Venkov [9] proved that these three conditions are sufficient for convex polytope to be a parallelohedron.

Centers of all tiles of the face-to-face tiling $\mathcal{T}(P)$ forms a d -dimension lattice $\Lambda(P)$. On the other hand for an arbitrary d -dimensional lattice Λ' we can construct a parallelohedron $P(\Lambda')$ which is a Dirichlet-Voronoi polytope for Λ' , i.e. the set of all points in \mathbb{R}^d that are closer to a given lattice point O than to any other point of Λ' .

Conjecture 1 (G. Voronoi, [10]). *Any d -dimensional parallelohedron P is affinely equivalent to a Dirichlet-Voronoi polytope $P(\Lambda')$ for some d -dimensional lattice Λ' .*

The conjecture of Voronoi was proved for several families of parallelohedra with local combinatorial properties.

Definition 1.1. We will denote the set of all k -faces of the tiling \mathcal{T} as \mathcal{T}^k , and the set of all k -faces of a given polytope P as P^k .

A face $F \in \mathcal{T}^{d-k}$ is called *primitive* if F is a face of exactly $k+1$ tiles of \mathcal{T} . For example any facet of \mathcal{T} is a primitive face because it belongs to exactly two tiles and for cubic tiling of \mathbb{R}^d only facets are primitive faces.

A face F of codimension 2 of the parallelohedron P is primitive if projection of P along F is a hexagon, otherwise if projection is a parallelogram then F is non-primitive and there are 4 copies of P in $\mathcal{T}(P)$ that contains F .

A parallelohedron P is called *k -primitive* if all k -faces of the tiling $\mathcal{T}(P)$ are primitive. A 0-primitive parallelohedron is also called a *primitive* parallelohedron. It is obvious that any k -primitive parallelohedron is a $(k+1)$ -primitive too.

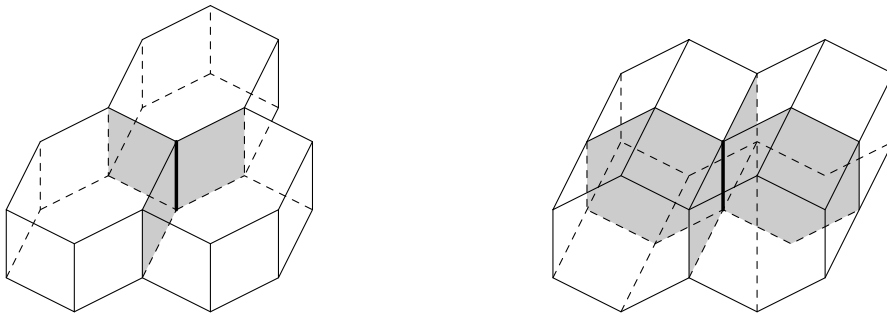


FIGURE 1. Local structure of primitive and non-primitive $(d - 2)$ -faces.

In 1909 Voronoi [10] proved his conjecture 1 for primitive parallelohedra, i.e. for the case when every vertex of the tiling $\mathcal{T}(P)$ belongs to exactly $d + 1$ copies of d -dimensional parallelohedron P . In 1929 Zhitomirskii [11] proved for $(d - 2)$ -primitive parallelohedra, i.e. for the case when every face of codimension 2 belongs to exactly 3 tiles. In this case all projections from Minkowski conditions are hexagons.

In 1999 Erdahl proved the Voronoi conjecture for space-filling zonotopes [2]. A *zonotope* is a Minkowski sum of several number of segments. Erdahl's proof significantly differs from results of Voronoi and Zhitomirskii above and result of Ordine below because it uses not the technique of positive quadratic forms but technique of unimodular vector representations.

For the moment the last case that was proved was done in 2005 by A. Ordine [7]. He proved the Voronoi conjecture for 3-irreducible parallelohedra. A parallelohedron P is called *k-irreducible* ($k > 1$) if for any face $F \in \mathcal{T}^{d-k}(P)$ the set of normals \mathcal{N}_F to facets of the tiling \mathcal{T} that contains F can not be represented as union of nonempty subsets \mathcal{N}_1 and \mathcal{N}_2 such that $\text{lin } \mathcal{N}_1 \cap \text{lin } \mathcal{N}_2 = \emptyset$. For example the result of Zhitomirskii [11] establishes the Voronoi conjecture for 2-irreducible parallelohedra. Also it is easy to prove that any k -irreducible parallelohedron is $(k + 1)$ -irreducible so Ordine's result covers both theorems of Voronoi and Zhitomirskii.

Definition 1.2. Consider the surface ∂P of the d -dimensional parallelohedron P . After deletion of all closed non-primitive faces of codimension 2 of P we will obtain a $(d - 1)$ -dimensional manifold without border. We will call this manifold the δ -surface of P and denote P_δ .

If we glue together every pair of opposite points of δ -surface of P then we will obtain another $(d - 1)$ -dimensional manifold that is subset of real projective space $\mathbb{R}P^{d-1}$. We will call this manifold the π -surface of P and denote P_π .

The δ -surface of P as well as P_π is connected if and only if P can not be represented as a direct sum of two parallelohedra of smaller dimensions [7]. Since it is enough to prove the Voronoi conjecture only for those parallelohedra that can not be represented as direct sums then further we will consider only parallelohedra with connected δ -surfaces.

In this paper we will prove (Theorem 4.3) the Voronoi conjecture for parallelohedra with simply connected δ -surface. Also we prove (Theorem 4.6) the Voronoi conjecture for a family of parallelohedra with fundamental group of π -surface generated by “trivial” cycles. These conditions are global while all preceeded conditions (Voronoi, Zhitomirskii and Ordine) were local. Our results generalize theorems of Voronoi and Zhitomirski but for now it is unclear whether our theorems are applicable Ordine's case or not.

2. CANONICAL SCALING AND SURFACE OF A PARALLELOHEDRON

This section is devoted to notion of canonical scaling and its connection with the proof of the Voronoi conjecture for a given parallelohedron P .

Definition 2.1. A function $\mathfrak{s} : \mathcal{T}^{d-1} \longrightarrow \mathbb{R}_+$ is called a *canonical scaling* for tiling \mathcal{T} or a parallelohedron P if for any $(d-2)$ -dimensional face $F \in \mathcal{T}^{d-2}$ we can choose directions of normals \mathbf{n}_i to all facets $F_i \in \mathcal{T}^{d-1}$ that contains F (here i ranges from 1 to 3 or 4 whether F is a primitive face or not) in such a way that

$$\sum_{i \text{ 3 or 4}} \mathfrak{s}(F_i) \mathbf{n}_i = \mathbf{0}. \quad (1)$$

Remark. Existence of a canonical scaling for a given tiling $\mathcal{T}(P)$ is equivalent to the Voronoi conjecture for parallelohedron P [3, 10]. This property was used in works of Voronoi [10], Zhitomirskii [11], and Ordine [7] to prove the Voronoi conjecture 1 for related polytopes.

Consider a primitive $(d-2)$ -face F . Then there are exactly 3 facets F_1, F_2, F_3 that contains F . Normals \mathbf{n}_i to these $(d-1)$ -faces spans a 2-dimensional plane since they are orthogonal to F and no two of them are linearly dependent, so there are exactly one (up to non-zero multiple) linear dependence

$$\alpha_1 \mathbf{n}_1 + \alpha_2 \mathbf{n}_2 + \alpha_3 \mathbf{n}_3 = \mathbf{0}.$$

Therefore if there exist a canonical scaling \mathfrak{s} for \mathcal{T} then

$$\frac{\mathfrak{s}(F_i)}{\mathfrak{s}(F_j)} = \left| \frac{\alpha_i}{\alpha_j} \right| \quad (2)$$

and if we want to construct a canonical scaling then we must obey this local rule.

Definition 2.2. The fraction $\left| \frac{\alpha_j}{\alpha_i} \right|$ we will call a *gain function* $\mathfrak{g}(F_i^{d-1}, F_j^{d-1})$. The gain function \mathfrak{g} is defined only on pairs of facets of the tiling with common primitive face of codimension 2 and if we switch arguments of the gain function then we will get a reciprocal number. The gain function \mathfrak{g} shows how the canonical scaling should change when we travel from one facet to another.

In order to prove that there is a canonical scaling it is enough to show that product of gain functions along any facet-cycle is 1. Here as a facet-cycle we consider a sequence of facet such that the first and the last one coincides and every two consecutive facets intersect in primitive $(d-2)$ -face of tiling \mathcal{T} .

Lemma 2.3. *If there exists a function $\mathfrak{s}' : P^{d-1} \longrightarrow \mathbb{R}_+$ that satisfies condition (2) for every two facets with common primitive $(d-2)$ -face then there exist a canonical scaling $\mathfrak{s} : \mathcal{T}^{d-1} \longrightarrow \mathbb{R}_+$.*

Remark. The backwards statement is trivial since we can take as \mathfrak{s}' the restriction of \mathfrak{s} on the surface of one copy of P .

Proof. Consider an arbitrary facet F and its opposite facet F' of P . If F belongs to some 6-belt then by applying rule (2) to this belt we will automatically get that $\mathfrak{s}'(F) = \mathfrak{s}'(F')$. If there is no such 6-belt then we change $\mathfrak{s}'(F')$ on the $\mathfrak{s}'(F)$ and this will does not break rule (2) for any $(d-2)$ -face of P .

Now we have a function \mathfrak{s}' that is invariant with respect to central symmetry of P . We translate this function on all copies of P in the tiling \mathcal{T} . This translation is correctly defined since in \mathcal{T} different tiles are glued by opposite facets of P and values of \mathfrak{s}' are

equal on these facets. Moreover the resulted function $\mathfrak{s} : \mathcal{T}^{d-1} \rightarrow \mathbb{R}_+$ is a canonical scaling because it satisfies condition (1) for every primitive $(d-2)$ -face because \mathfrak{s}' satisfies rule (2) and it satisfies condition (1) for non primitive $(d-2)$ -face because every face like this lies in two pairs of opposite facets and we can choose opposite normal directions for facets in one pair. \square

Definition 2.4. We will say that sequence of facets $\gamma = [F_0, \dots, F_k]$ is a *primitive combinatorial path* of P if every two consequent facets F_i and F_{i+1} has a common primitive face of codimension 2 of P . We call γ a *primitive cycle* if $F_0 = F_k$.

Now we can define the gain function \mathfrak{g} for every primitive path of P by the formula

$$\mathfrak{g}(\gamma) = \prod_{i=1}^k \mathfrak{g}(F_{i-1}, F_i).$$

We will call a curve γ on the δ -surface of P (see Definition 1.2) *generic* if endpoints of γ are in the interior of facets of P , γ does not intersect any face of dimension less than $d-2$, and any intersection of γ with $(d-2)$ -dimensional face of P is transversal.

Now for every generic curve γ on the δ -surface of P we can write down a *supporting primitive path* γ' from facets that supports γ and we can define the gain function $\mathfrak{g}(\gamma) := \mathfrak{g}(\gamma')$. It is evident that if a curve is a union of two curves $\gamma = \gamma_1 \cup \gamma_2$ then $\mathfrak{g}(\gamma) = \mathfrak{g}(\gamma_1) \cdot \mathfrak{g}(\gamma_2)$.

Lemma 2.5. *The Voronoi conjecture 1 is true for the parallelohedron P if and only if for every generic cycle γ on the δ -surface of P we have $\mathfrak{g}(\gamma) = 1$.*

Proof. If for every generic cycle γ we have $\mathfrak{g}(\gamma) = 1$ then we can construct a function $\mathfrak{s}' : P^{d-1} \rightarrow \mathbb{R}_+$ from the lemma in the following way. Consider an arbitrary facet F and put $\mathfrak{s}'(F) = 1$. Now for every facet G consider an arbitrary generic curve γ that starts in the center of F and ends in the center of G and put $\mathfrak{s}'(G) = \mathfrak{g}(\gamma)$. We will show that this function \mathfrak{s}' is defined correctly and that it satisfies conditions of the lemma 2.3.

Assume that for two different curves γ_1 and γ_2 we will obtain different values of \mathfrak{s}' on the facet G so $\mathfrak{g}(\gamma_1) \neq \mathfrak{g}(\gamma_2)$ then for cycle $\gamma = \gamma_1 \cup \gamma_2^{-1}$ (here γ_2^{-1} denotes the inverse curve of the γ_2) we will have $\mathfrak{g}(\gamma) = \frac{\mathfrak{g}(\gamma_1)}{\mathfrak{g}(\gamma_2)} \neq 1$ and this contradicts with conditions of this lemma.

In order to show that \mathfrak{s}' satisfies conditions of the lemma 2.3 consider two arbitrary facets F_1 and F_2 with common non-primitive $(d-2)$ -face. Assume that values on facets F_i were obtained with paths γ_i , consider a path γ_3 that connects centers of F_1 and F_2 through their common $(d-2)$ -face. We have the cycle $\gamma = \gamma_3 \cup \gamma_2^{-1} \cup \gamma_1$ so

$$\mathfrak{g}(\gamma_3) = \frac{\mathfrak{g}(\gamma_2)}{\mathfrak{g}(\gamma_1)} = \frac{\mathfrak{s}'(F_2)}{\mathfrak{s}'(F_1)}$$

but by definition $\mathfrak{g}(\gamma_3) = \mathfrak{g}(F_1, F_2)$ and therefore \mathfrak{s}' satisfies condition of the lemma 2.3 and the Voronoi conjecture is true for P .

On the other hand, if the Voronoi conjecture is true for P then there is a canonical scaling $\mathfrak{s} : \mathcal{T}^{d-1} \rightarrow \mathbb{R}_+$ and our gain function $\mathfrak{g}(F, G)$ coincides with fraction of values of \mathfrak{s} on the facets G and F so it is clear that for every generic cycle γ on the δ -surface of P we have $\mathfrak{g}(\gamma) = 1$. \square

3. DUAL 3-CELLS AND LOCAL CONSISTENCY OF CANONICAL SCALING

Definition 3.1. Consider an arbitrary face F of codimension k of the tiling $\mathcal{T}(P)$. Then centers of all copies of P that shares F forms a *dual k -cell* \mathcal{D}_F correspondent to F . For

example, the dual cell correspondent to a d -dimensional polytope $P' \in \mathcal{T}(P)$ is a single point — its center.

For dual cells we can introduce the notion of incidence. Namely, dual cell \mathcal{D}_F is incident to a dual cell \mathcal{D}_G if and only if G is incident to F . With this notion of incidence the set of all dual cells of the tiling $\mathcal{T}(P)$ that contains in a given dual k -cell \mathcal{D}_F gives us a face lattice that “looks like” a face lattice of k -dimensional polytope. So we can say that dual k -cell is *combinatorially equivalent* to k -dimensional polytope with the same face lattice (if such a polytope exist). For example dual 2-cells are equivalent to triangle or parallelogram for primitive or non-primitive correspondent faces of codimension two (see figure 1).

Conjecture 2. *Convex hull of points of dual k -cell \mathcal{D}_F is a k -dimensional polytope. Moreover the set of all dual cells gives us a tiling dual to $\mathcal{T}(P)$.*

Remark. If tiling $\mathcal{T}(P)$ is a Dirichlet-Voronoi tiling then there exist a dual Delone tiling [8], so conjecture 2 is necessary to Voronoi conjecture 1.

We will use the following classification of dual 3-cells.

Theorem 3.2 (Delone, [1]). *There are five different types of possible dual 3-cells, namely they are combinatorially equivalent to the following 3-dimensional polytopes: cube, triangular prism, tetrahedron, octahedron, and quadrangular prism.*

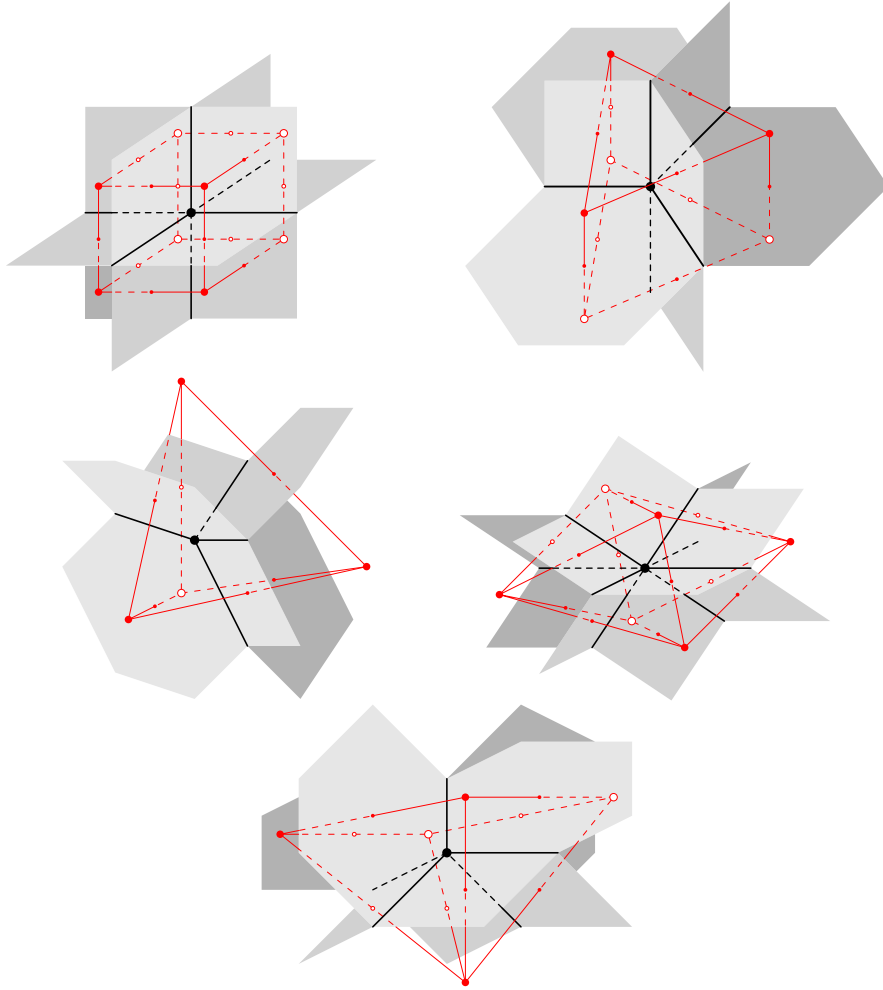


FIGURE 2. Dual 3-cells.

Lemma 3.3. *Given $(d-3)$ -face F of d -dimensional parallelhedron P . If F belongs only to primitive $(d-2)$ -faces of P then gain function \mathbf{g} is equal to 1 on the primitive cycle around F .*

Remark. By cycle around F we mean the cycle that passes every facet incident to F once, this cycle is defined correctly since the surface of P is $(d-1)$ -dimensional manifold, so there is exactly two-dimensional submanifold that transversal to F and we did not delete any point from neighbourhood of F because F does not belong to non-primitive $(d-2)$ -faces of P .

If we take the correspondet dual 3-cell of F then this cell can be combinatorially equivalent to tetrahedron, octahedron or quadrangular pyramid. In the first two cases P can be represented by any vertex of dual cell but in the latter case P can be represented only by the apex of pyramid since only in these cases all 2-dimensional faces of dual cell incident to P are triangular.

Throughout the proof we will not focus on dimensional aspects that arises from local combinatorics of the tiling into copies of P . One can easily check that all objects that we are treating as three-dimensional are three-dimensional.

Proof. It is obvious that the Dimension conjecture 2 is true for dual 3-cells and we will use it further. Moreover, any dual 3-cell is one of 3 three-dimensional polytopes (metrically!): a tetrahedron, an octahedron (not necessarily regular) or a pyramid over parallelogram. We will consider all three cases of possible dual 3-cells of F .

Denote the cycle around F on the surface of P as γ .

First, assume that dual 3-cell of F is a tetrahedron \mathcal{F} with vertices correspondent to parallelhedra P_0, P_1, P_2 and P_3 ($P = P_0$). Denote centers of these polytopes as A_0, A_1, A_2, A_3 respectively, these points will be exactly four vertices of \mathcal{F} . Then F belongs to exactly 6 facets $F_{ij} = F_{ji}, 0 \leq i, j \leq 3, i \neq j$ of the tiling, F_{ij} is the common facet of polytopes P_i and P_j . And there are 4 faces of codimension 2 of the tiling that are incident to F . We need to show that $\mathbf{g}(\gamma) = \mathbf{g}([F_{01}, F_{02}, F_{03}, F_{01}]) = 1$.

The 3-dimensional plane $A_0A_1A_2A_3$ is transversal to the face F so we can find an appropriate affine transformation \mathcal{A} such that the image of this plane will be orthogonal to the image of F . Applying any affine transformation does not change the property that $\mathbf{g}(\gamma) = 1$.

Now if we orthogonally project our four parallelhedra P_0, \dots, P_3 onto the plane of tetrahedron $A_0A_1A_2A_3$ then the face F will project into a point F' . Let \mathbf{e}_i be the vector of projection of $(d-2)$ -dimensional face of the tiling that is common to all parallelhedra incident to F but P_i . Four vectors \mathbf{e}_i generates three-dimensional subspace parallel to $A_0A_1A_2A_3$ and any three of these vectors are linearly independent, so upto homothety there is exactly one tetrahedron $\mathcal{F}^* = B_0B_1B_2B_3$ in the space $A_0A_1A_2A_3$ such that each vector \mathbf{e}_i is the normal vector to the facet of \mathcal{F}^* that does not contain vertex B_i .

We claim that the gain function \mathbf{g} and therefore canonical scaling is defined by lengths of edges of tetrahedron \mathcal{F}^* . Namely, $\mathbf{g}(F_{ij}, F_{ik}) = \frac{|B_iB_k|}{|B_iB_j|}$. Indeed, the segment B_1B_2 is orthogonal both faces $P_1 \cap P_2 \cap P_3$ and $P_1 \cap P_2 \cap P_4$, hence it is orthogonal to the facet $F_{12} = P_1 \cap P_2$. By the same argument any segment B_iB_j is orthogonal to the face F_{ij} . Therefore each facet of \mathcal{F}^* represents the unique linear dependence of normal vectors of facets of the tiling incident to a primitive face of codimension 2. So,

$$\mathbf{g}(F_{01}F_{02}F_{03}F_{01}) = \mathbf{g}(F_{01}, F_{02}) \cdot \mathbf{g}(F_{02}, F_{03}) \cdot \mathbf{g}(F_{03}, F_{01}) = \frac{|B_0B_2|}{|B_0B_1|} \cdot \frac{|B_0B_3|}{|B_0B_2|} \cdot \frac{|B_0B_1|}{|B_0B_3|} = 1.$$

And the first case is done.

Now assume that dual face of F is an octahedron \mathcal{F} with vertices, correspondent to parallelohedra P, Q, R, P', Q', R' , here prime denote the vertex centrally symmetric to the same without prime. We will denote vertices of \mathcal{F} with the same letters as correspondent polytopes. Then F belongs to twelve facets of the tiling correspondent to edges $PQ, PR, PQ', PR', P'Q, P'R, P'Q', P'R', QR, QR', Q'R, Q'R'$ and to eight $(d-2)$ -faces of the tiling correspondent to triangular faces $PQR, PQR', PQ'R, PQ'R', P'QR, P'QR', P'Q'R, P'Q'R'$ of the octahedron \mathcal{F} . As before we will denote as F_{ab} the facet common to parallelohedra a and b . So, we need to show that $\mathbf{g}(\gamma) = \mathbf{g}([F_{PQ}, F_{PR}, F_{PQ'}, F_{PR'}]) = 1$.

As before we can find suitable affine transformation and orthogonal projection on the 3-dimensional plane of octahedron \mathcal{F} . The projection of the face F is the center of this octahedron since this center is the center of symmetry of the lattice of the tiling that interchanges P and P' . Moreover, the octahedron \mathcal{F} is centrally symmetric with respect to projection of F . Let \mathbf{e}_{PQR} denotes the vector of projection of $(d-2)$ -dimensional face correspondent to dual 2-cell PQR . In the same way we will denote vectors of projections of other 7 $(d-2)$ -faces. We can assume that $\mathbf{e}_{PQR} = -\mathbf{e}_{P'Q'R'}$ and similarly for other 3 pairs of vectors.

Like in the previous case we will construct a polytope such that its edges will give us normals of facets incident to F . But unlike in previous case we will use only part of constructed vectors since there are four pairs of opposite between them. Let $ABCD$ be a tetrahedron whose facet are orthogonal to constructed vectors, namely, ABC is orthogonal to \mathbf{e}_{PQR} , ABD is orthogonal to $\mathbf{e}_{P'Q'R'}$, ACD is orthogonal to $\mathbf{e}_{P'Q'R}$, and BCD is orthogonal to $\mathbf{e}_{PQ'R'}$. Such a tetrahedron is unique upto homothety.

The edges of $ABCD$ are orthogonal to the facets of the tiling incident to F . For example, the edge AB is the intersection of two faces ABC and ABD . The first one is orthogonal to \mathbf{e}_{PQR} and therefore $\mathbf{e}_{P'Q'R'}$ and the second one is orthogonal to $\mathbf{e}_{P'Q'R}$ and $\mathbf{e}_{PQ'R'}$. So AB is orthogonal to all four these vectors and it is parallel to the projection of normal vector to facets F_{PR} and $F_{P'R'}$ of the tiling. In the same way we can prove that any edge of tetrahedron $ABCD$ is parallel to normal of two facets incident to F . In particularly BC is parallel to normal vectors of F_{QR} and $F_{Q'R'}$ and AC is parallel to normal vectors of F_{PQ} and $F_{P'Q'}$. Now we see that edges of triangle ABC represents the unique linear dependence between normals of common facets of polytopes P, Q and R , so $\mathbf{g}(F_{PQ}, F_{QR}) = \frac{|BC|}{|AC|}$. Similarly we can find the value of gain function \mathbf{g} on any to pairs of facets incident to F (except opposite) using only lengths of edges of $ABCD$. And finally we get

$$\begin{aligned} \mathbf{g}(\gamma) &= \mathbf{g}(F_{PQ}, F_{PR}) \cdot \mathbf{g}(F_{PR}, F_{PQ'}) \cdot \mathbf{g}(F_{PQ'}, F_{PR'}) \cdot \mathbf{g}(F_{PR'}, F_{PQ}) = \\ &= \frac{|AB|}{|AC|} \cdot \frac{|BD|}{|AB|} \cdot \frac{|CD|}{|BD|} \cdot \frac{|AC|}{|CD|} = 1. \end{aligned}$$

The second case is done.

Now we have only the last case left, namely assume that dual 3-cell for F is quadrangular pyramid \mathcal{F} with base $QRQ'R'$ and apex P . Here vertices of the dual cell represents parallelohedra of the tiling as before. We will use similar notations as before and the general idea to prove the consistence of the cycle $\gamma = [F_{PQ}, F_{PR}, F_{PQ'}, F_{PR'}, F_{PQ}]$ will be the same: we will construct a three-dimensional polytope with edges parallel to normal vectors to facets incident to F .

After affine transformation and orthogonal projection onto the plane of \mathcal{F} $(d-2)$ -faces of the tiling incident to F will be represented by five vectors correspondent to faces of \mathcal{F} . Namely $\mathbf{e}_{PQR}, \mathbf{e}_{PQR'}, \mathbf{e}_{PQ'R}, \mathbf{e}_{PQ'R'}$ which correspond to triangular faces of \mathcal{F} , and $\mathbf{e}_{QRQ'R'}$ which corresponds to the base of \mathcal{F} .

The only difference in that case is in the constructing desired polytope. Since we have 5 vectors that should be orthogonal to its faces, there are two possibilities of such polytope (combinatorial triangular prism or quadrangular pyramid) and only one of these cases will lead us to gain function. To avoid undesired case we will take an arbitrary point O in the space of pyramid \mathcal{F} and take four planes in this space that are orthogonal to considered vectors but $\mathbf{e}_{QRQ'R'}$. To finish the construction we take a plane that is orthogonal to $\mathbf{e}_{QRQ'R'}$ and do not contain O . We had constructed the quadrangular pyramid $OABCD$ with apex O such that its faces OAB , OBC , OCD , ODA , and $ABCD$ are orthogonal to vectors \mathbf{e}_{PQR} , $\mathbf{e}_{PQ'R}$, $\mathbf{e}_{PQ'R'}$, \mathbf{e}_{PQR} , and $\mathbf{e}_{QRQ'R'}$ respectively.

And again edges of constructed pyramed are parallel to normals to facets incident to F . So we can write down a formula for $\mathfrak{g}(\gamma)$:

$$\begin{aligned}\mathfrak{g}(\gamma) &= \mathfrak{g}(F_{PQ}, F_{PR}) \cdot \mathfrak{g}(F_{PR}, F_{PQ'}) \cdot \mathfrak{g}(F_{PQ'}, F_{PR'}) \cdot \mathfrak{g}(F_{PR'}, F_{PQ}) = \\ &= \frac{|OD|}{|OA|} \cdot \frac{|OC|}{|OD|} \cdot \frac{|OB|}{|OC|} \cdot \frac{|OA|}{|OB|} = 1.\end{aligned}$$

□

4. THE VORONOI CONJECTURE FOR PARALLELOHEDRA WITH SIMPLY CONNECTED δ -SURFACE

Lemma 4.1. *If two generic cycles γ_1 and γ_2 on the δ -surface of parallelohedron P are homotopy equivalent then $\mathfrak{g}(\gamma_1) = \mathfrak{g}(\gamma_2)$.*

Remark. Under “homotopy” here and after we mean the relation of continuous homotopy.

Proof. Consider an arbitrary homotopy $F(t)$ between γ_1 and γ_2 such that $F(0) = \gamma_1$ and $F(1) = \gamma_2$. With small perturbation of the homotopy F we can obtain another homotopy $G(t)$ such that:

- (1) $G(0) = \gamma_1$ and $G(1) = \gamma_2$;
- (2) at any time t cycle $G(t)$ does not intersect any face of P with dimension less than $d - 3$;
- (3) at any time t cycle $G(t)$ does not have more than one points of intersection with $(d - 3)$ -faces of P ;
- (4) there are only finite number of times t_1, \dots, t_n such that each cycle $G(t_i)$ intersects some $(d - 3)$ -face F_i^{d-3} of P ;
- (5) there are only finite number of times τ_1, \dots, τ_k ($t_i \neq \tau_j$) such that for each cycle $G(\tau_j)$ there is exactly one non-transversal intersection of $G(\tau_j)$ with some $(d - 2)$ -face F_j^{d-2} . For all other $t \neq \tau_j$ each intersection of $G(t)$ with $(d - 2)$ -faces of P is transversal.

So in other words we can find a perturbation of the homotopy $F(t)$ that will be in general position.

For any time t that is not equal to any t_i and any τ_j the cycle $G(t)$ is generic so we can define the gain function $\mathfrak{g}(G(t))$. We will show that this gain function does not depend on t .

For any segment $[a, b] \subset [0, 1]$ that does not contain points t_i and τ_j the gain function $\mathfrak{g}(G(t))$ is constant because for all $t \in [a, b]$ primitive path γ'_t that supports $G(t)$ does not depend on t . So the only thing that we need to show is that gain function $\mathfrak{g}(G(t))$ does not change when we pass across t_i or across τ_j .

When t passes across τ_j then the supporting primitive path does not changes or one of its facets F^{d-1} could be replaced by several copies of sequence $[F^{d-1}, G^{d-1}, F^{d-1}]$ or vice

versa for some facets F^{d-1} and G^{d-1} with common primitive $(d-2)$ -face. In that case gain function \mathbf{g} will not change because $\mathbf{g}([F^{d-1}, G^{d-1}, F^{d-1}]) = 1$.

When t passes across t_i then the supporting primitive path does not change or some subpath $[F_{i,1}^{d-1}, \dots, F_{i,2}^{d-1}]$ such that every facet of this path contains F_i^{d-3} changes into subpath with the same startpoint and endpoint and again all facets of the new subpath will contain F_i^{d-3} . In that case the gain function $\mathbf{g}(G(t))$ will not change due to lemma 3.3.

So the gain function $\mathbf{g}(F(t))$ is constant and $\mathbf{g}(\gamma_1) = \mathbf{g}(F(0)) = \mathbf{g}(F(1)) = \mathbf{g}(\gamma_2)$. \square

Corollary 4.2. *The gain function \mathbf{g} is a homomorphism of fundamental group $\pi_1(P_\delta)$ into \mathbb{R}_+ .*

Since \mathbb{R}_+ is commutative group than we trivially get that \mathbf{g} also gives us a homomorphism of group $\pi_1(P_\delta)/[\pi_1(P_\delta)]$ (here $[G]$ denotes the commutant of the group G) to \mathbb{R}_+ . This group is isomorphic to group of homologies $H_1(P_\delta)$ (see [4]).

Theorem 4.3. *Given a parallelohedron P with connected δ -surface. If fundamental group $\pi_1(P_\delta)$ or group of homologies $H_1(P_\delta)$ is trivial then the Voronoi conjecture 1 is true for P .*

Proof. In both cases an arbitrary generic cycle γ can be represented as a product $\gamma = \gamma_1 \dots \gamma_k$ where $\gamma_i = a_i b_i a_i^{-1} b_i^{-1}$ is a cycle from commutant $[\pi_1(P_\delta)]$. This is true because in both cases $[\pi_1(P_\delta)] = \pi_1(P_\delta)$ (and also this is trivial group in the first case). It is clear that $\mathbf{g}(\gamma_i) = 1$ and hence $\mathbf{g}(\gamma) = 1$. Thus by lemma 2.5 the Voronoi conjecture is true for P . \square

Now we will show how to generalize this theorem for the case of π -surface of parallelohedron P . This theorem can be generalized in two directions. First we can consider polytopes with non-connected δ - or π -surfaces. In that case the polytope P can be represented as a direct sum of parallelohedra of smaller dimensions as it was proved in [7], so for both polytope summands the theorem 4.3 and therefore the Voronoi conjecture is true so it is true for P .

The second way of generalizing theorem 4.3 is to introduce new cycles that also have gain function 1. On the π -surface of P such cycles are “half-belt” cycles.

Definition 4.4. The cycle γ on P_π is called *half-belt cycle* if its support is a combinatorial path $\gamma' = [F_1, F_2, F_3, F_1]$ such that all three facets F_i belongs to the same belt of length 6.

On the δ -surface P_δ this cycle corresponds to a path that starts on the facet F_1 and ends on the opposite facet F'_1 of P and crossing only three parallel primitive faces of codimension 2.

Lemma 4.5. *For every half-belt cycle γ we have $\mathbf{g}(\gamma) = 1$.*

Proof. Let $\alpha_1 \mathbf{n}_1 + \alpha_2 \mathbf{n}_2 + \alpha_3 \mathbf{n}_3 = \mathbf{0}$ be the unique linear dependence of normal vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ to facets F_1, F_2 , and F_3 . Then by definition of the gain function \mathbf{g} we have

$$\mathbf{g}(\gamma) = \mathbf{g}(F_1, F_2) \cdot \mathbf{g}(F_2, F_3) \cdot \mathbf{g}(F_3, F'_1) = \frac{|\alpha_2|}{|\alpha_1|} \cdot \frac{|\alpha_3|}{|\alpha_2|} \cdot \frac{|\alpha_1|}{|\alpha_3|} = 1.$$

\square

Theorem 4.6. *If the fundamental group $\pi_1(P_\pi)$ or the group of homologies $H_1(P_\pi)$ is generated by half-belt cycles of parallelohedron P then the Voronoi conjecture is true for P .*

Proof. We can use the same arguments as in the proof of theorem 4.3. As before we can show that the gain function \mathbf{g} is a homomorphism of $\pi_1(P_\pi)$ to \mathbb{R}_+ . We only need to add that since any cycle γ can be represented as a product $\gamma = \prod \gamma_i$ of half-belt cycles γ_i then $\mathbf{g}(\gamma) = 1$ as desired. \square

Remark. When we factorize a fundamental group of a manifold M by its commutant we will get the \mathbb{Z} -homologies of M . But in our case it can be easily generalized to the case of \mathbb{Q} -homologies. Moreover we are only interested in the torsion-free part of homology group and in the case of \mathbb{Q} -homologies this property is automatically satisfied.

5. THREE-DIMENSIONAL CASE

In this section we will show that conditions of theorem 4.6 are true for all three-dimensional parallelohedra.

Example 5.1. There are five combinatorial types of three-dimensional parallelohedra. Two reducible parallelohedra are cube \mathcal{C} and hexagonal prism \mathcal{P} . It is easy to see that \mathcal{C}_δ is a collection of six disjoint open disks (correspondent to faces of \mathcal{C}) and \mathcal{C}_π is a collection of three disjoint open disks. In both cases it is easy to see that both fundamental groups $\pi_1(\mathcal{C}_\delta)$ and $\pi_1(\mathcal{C}_\pi)$ as well as homology groups $H_1(\mathcal{C}_\delta)$ and $H_1(\mathcal{C}_\pi)$ are trivial and conditions of both theorems 4.3 (for non-connected case) and 4.6 are true. For \mathcal{P} situation is a bit more interesting. So, \mathcal{P}_δ is a collection of two open disks (bases of prism) and an open strip (the side surface) and \mathcal{P}_π is a collection of a disk and a Möbius strip. And in that case

$$\pi_1(\mathcal{P}_\delta) \cong \pi_1(\mathcal{P}_\pi) \cong H_1(\mathcal{P}_\delta) \cong H_1(\mathcal{P}_\pi) \cong \mathbb{Z}.$$

It is easy to find generators for all these groups. For both fundamental and homology group of δ -surface the generator is the cycle represented by single belt-cycle on side surface of \mathcal{P} . End for π -surface generators are represented by unique half-belt cycle of \mathcal{P} .

Three irreducible polytopes are rhombic dodecahedron \mathcal{R} , elongated dodecahedron \mathcal{E} , and truncated octahedron \mathcal{O} (from left to right on the next figure). The rhombic dodeca-

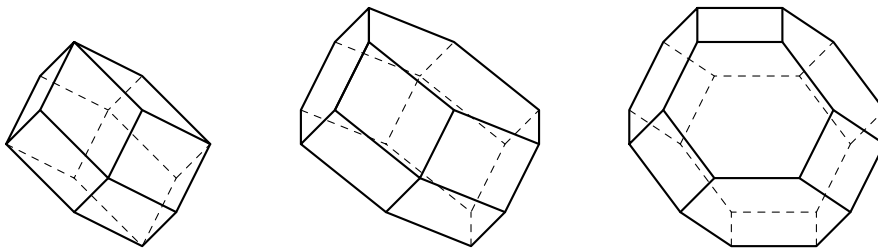


FIGURE 3. Irreducible parallelohedra in \mathbb{R}^3 .

hedron and truncated octahedron are 2-primitive (the Zhitomirskii case [11]) since all its edges generates belts of length 6. For such parallelohedra δ -surface is just the surface of parallelohedra itself. So, \mathcal{R}_δ and \mathcal{O}_δ are homeomorphic to sphere \mathbb{S}^2 , and \mathcal{R}_π and \mathcal{O}_π are homeomorphic to projective plane \mathbb{RP}^2 . So, groups $\pi_1(\mathcal{R}_\delta)$, $H_1(\mathcal{R}_\delta)$, $\pi_1(\mathcal{O}_\delta)$ and $H_1(\mathcal{O}_\delta)$ are trivial and we can apply theorem 4.3 for these polytopes. And for π -surfaces we know that groups $\pi_1(\mathcal{R}_\pi)$, $H_1(\mathcal{R}_\pi, \mathbb{Z})$, $\pi_1(\mathcal{O}_\pi)$ and $H_1(\mathcal{O}_\pi, \mathbb{Z})$ are isomorphic to \mathbb{Z}_2 and they are generated by any half-belt cycle so we can apply theorem 4.6 also. Also, as we discussed in the remark after theorem 4.6 we can consider \mathbb{Q} -homologies also. For both parallelohedra \mathcal{R} and \mathcal{O} correspondent \mathbb{Q} -homologies are trivial so we can apply the theorem 4.6 for this case also.

And the last and the most interesting case is the elongated dodecahedron \mathcal{E} . Here the manifold \mathcal{E}_δ is a sphere with four cuts correspondent to four vertical edges of the middle polytope of the figure 3. And the fundamental group $\pi_1(\mathcal{E}_\delta)$ is the free group on three letters. We can take three cycles consist of two half-belts each around some edges that we deleted as independent generators. And the homology group $H_1(\mathcal{E}_\delta)$ is isomorphic to \mathbb{Z}^3 .

The manifold \mathcal{E}_π is the real projective plane with two cuts. So, fundamental group $\pi_1(\mathcal{E}_\pi)$ is generated by two cycles around cuts and arbitrary half-belt cycle but these three cycles are not independent. Groups of first homologies also can be found easily using that Betti number h_1 is 1 (to check this one can calculate the Euler characteristic of \mathcal{E}_π). For \mathbb{Q} -homologies we have $H_1(\mathcal{E}_\pi, \mathbb{Z}) = \mathbb{Z} \times T$, where T is the torsion part of homologies group and does not affect on existence of canonical scaling. And $H_1(\mathcal{E}_\pi, \mathbb{Q})$ is just \mathbb{Q} . And in both case non-torsion part of group can be represented by cycle consist of two half-belt cycles around one of two cuts. So theorem 4.6 can be applied in this case also.

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